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A (p, q)-deformed Virasoro algebra

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Abstract. A conformal dimension (Δ) dependent (p, q)-deformed Virasoro ((p, q)-Virasoro) algebra with two independent deformation parameters (p, q) is constructed. The comultiplication rule for the generating functional for $\Delta = 0, 1$ case is established and found to be depending on p and q individually. The central charge term for the (p, q)-Virasoro algebra is described. A (p, q)-deformed nonlinear equation ((p, q)-KdV) corresponding to (p, q)-Virasoro algebra for $\Delta = 0, 1$ case is obtained.

1. Introduction

Recently there have been many attempts [1–7] towards finding a quantum deformation of Virasoro algebra. Considering algebras with a single deformation parameter (q-Virasoro), these authors studied the multiplicative structure, the comultiplication rule for the deformed generators [6], the deformation of the central extension term [3, 5, 7] and the deformed Korteweg-de Vries (q-KdV) equation [4] corresponding to the q-Virasoro algebra. In particular, Chaichian *et al* [7] considered a q-deformation of the differential realization of the centreless Virasoro algebra and obtained a conformal dimension (Δ) dependent deformed structure which satisfied a deformed Jacobi identity. This requirement led to a central extension of the deformed Virasoro algebra compatible with the conventional centre in the $q \to 1$ limit.

In an alternate development, the construction and the representation theory for the quantum groups and algebras with multiple deformation parameters have been studied [8–15]. Based on an oscillator realization with two independent parameters (p, q) the present authors constructed [14] a centreless (p, q)-deformed Virasoro ((p, q)-Virasoro) algebra. Using the results of (p, q)-analysis developed therein, here we obtain, *à la* Chaichian *et al* [7], a Δ-dependent differential realization of the (p, q)-Virasoro algebra. This algebra satisfies a (p, q)-deformed Jacobi identity and thereby a central extension term for the algebra may be established. The centreless deformed (p, q)-Virasoro algebra, we find, depends essentially on one parameter $(Q = \sqrt{pq})$. However, as emphasized by Schirrmacher *et al* [13] in another context, whether (p, q) are two genuinely independent parameters, is to be settled by the structure of the comultiplication rule. Following Devchand *et al* [6] we use a continuum formulation for the (p, q)-Virasoro algebra in the $\Delta = 0, 1$ case to obtain a comultiplication rule for the (p, q)-deformed generating functional, which truly depends on both parameters. For a generic Δ ($\neq 0, 1$), we could not obtain the relevant comultiplication rule, but, it is natural to assume that the above-mentioned qualitative feature for $\Delta = 0, 1$

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case would persist for an arbitrary Δ. Further, the central extension term is found to depend on p and q individually, and not necessarily Q alone, showing that p and q are two genuinely independent deformation parameters.

The close kinship [16, 17] between the Virasoro algebra and the KdV equation was followed by Chaichian *et al* [4] to obtain a q-deformed KdV equation corresponding to the q-Virasoro algebra. A similar treatment links our (p, q)-Virasoro algebra (3.21) for the $\Delta=0, 1$ case with a (p, q)-deformed nonlinear structure ((p, q)-KdV), which in the undeformed limit $(p, q \to 1)$ reduces to the usual KdV equation. The constructions of a (p, q)-Virasoro algebra for an arbitrary Δ $(\neq 0, 1)$ and for $\Delta=0, 1$ cases are described in sections 2 and 3 respectively. We conclude in section 4.

2. (p, q)-Virasoro algebra for an arbitrary Δ $(\neq 0, 1)$

To construct a (p, q)-Virasoro algebra we closely follow the well known route for the undeformed case. An arbitrary primary field $\phi_\Delta(z)$ with the conformal dimension Δ transforms under an infinitesimal coordinate transformation

$$z \to z + \varepsilon(z) \tag{2.1}$$

as

$$\delta_{\varepsilon(z)}\phi_\Delta(z) = \varepsilon(z)^{1-\Delta}\partial_z(\varepsilon(z)^\Delta\phi_\Delta(z)). \tag{2.2}$$

For the choice

$$\varepsilon(z) = z^{n+1} \tag{2.3}$$

we obtain

$$\delta_n\phi_\Delta(z) \equiv \ell_n\phi_\Delta(z) = (z\partial_z + \Delta(n+1) - n)z^n\phi_\Delta(z) \tag{2.4}$$

where the generators ℓ_n satisfy the centreless Virasoro algebra

$$[\ell_n, \ell_m] = (m-n)\ell_{n+m}. \tag{2.5}$$

Following the above strategy, we define the infinitesimal (p, q)-transformation for the primary field $\phi_\Delta(z)$ as

$$\delta^{p,q}_{\varepsilon(z)}\phi_\Delta(z) = \varepsilon(z)^{1-\Delta}D_{p,q}(\varepsilon(z)^\Delta\phi_\Delta(z)) \tag{2.6}$$

where the (p, q)-deformed derivative is given [14] by

$$D_{p,q}f(z) = \frac{f(qz) - f(p^{-1}z)}{z(q-p^{-1})} = z^{-1}[z\partial_z]f(z) \tag{2.7}$$

and

$$[x] = \frac{q^x - p^{-x}}{q-p^{-1}}. \tag{2.8}$$

For the choice (2.3) we define the (p, q)-Virasoro generators $L_n^{(\Delta)}$ by

$$\delta_n^{p,q}\phi_\Delta(z) \equiv L_n^{(\Delta)}\phi_{(\Delta)}(z) = [z\partial_z + \Delta(n+1) - n]z^n\phi_\Delta(z). \tag{2.9}$$

For an arbitrary Δ $(\neq 0, 1)$ the generators $L_n^{(\Delta)}$ satisfy a closed algebraic structure

$$[L_n^{(\Delta)}, L_m^{(\Delta)}]_{x_\Delta, y_\Delta} = (q-p^{-1})^{-1}\{q^{N_\Delta}(x_\Delta q^{-n} - y_\Delta q^{-m}) - p^{-N_\Delta}(x_\Delta p^n - y_\Delta p^m)\}L_{n+m}^{(\Delta)} \tag{2.10}$$

where

$$[A, B]_{x,y} = xAB - yBA \tag{2.11}$$

$$N_\Delta = z\partial_z + \Delta \tag{2.12}$$

$$[N_\Delta, L_n^{(\Delta)}] = nL_n^{(\Delta)} \tag{2.13}$$

$$x_\Delta = \left(\frac{q}{p}\right)^n \frac{[n(\Delta-1)][\Delta m]}{[n][m]} \tag{2.14}$$

$$y_\Delta = \left(\frac{q}{p}\right)^m \frac{[m(\Delta-1)][\Delta n]}{[n][m]}. \tag{2.15}$$

The construct (2.10), in the limit $p = q$, agrees with the q-Virasoro algebra obtained by Chaichian *et al* [7]. The algebra (2.10) may be regarded as a (p, q)-Virasoro algebra for an arbitrary Δ ($\neq 0, 1$) and reduces to (2.5) in the limit ($p, q \to 1$). The operator-valued structure constants in (2.10) depend on the conformal dimension Δ and the generator of the scale transformation $L_0^{(\Delta)}$ ($=[N_\Delta]$). With a redefinition of the parameters

$$Q = \sqrt{pq} \qquad \lambda = \sqrt{p/q} \tag{2.16}$$

the algebra (2.10) may, however, be mapped to the Q-Virasoro algebra studied by Chaichian *et al* [7]. The mapping is given by

$$\mathscr{L}_n^{(\Delta)} = \lambda^{N_\Delta - 1} L_n^{(\Delta)}. \tag{2.17}$$

The generators $\mathscr{L}_n^{(\Delta)}$ satisfy the algebra

$$\begin{aligned}[\mathscr{L}_n^{(\Delta)}, \mathscr{L}_m^{(\Delta)}]_{x_\Delta^Q, y_\Delta^Q} &= (Q - Q^{-1})^{-1}\{Q^{N_\Delta}(x_\Delta^Q Q^{-n} - y_\Delta^Q Q^{-m}) \\ &\quad - Q^{-N_\Delta}(x_\Delta^Q Q^n - y_\Delta^Q Q^m)\}\mathscr{L}_{n+m}^{(\Delta)}\end{aligned} \tag{2.18}$$

where

$$x_\Delta^Q = \frac{[n(\Delta-1)]_Q[\Delta m]_Q}{[n]_Q[m]_Q} \tag{2.19}$$

$$y_\Delta^Q = \frac{[m(\Delta-1)]_Q[\Delta n]_Q}{[n]_Q[m]_Q} \tag{2.20}$$

$$[x]_Q = \frac{Q^x - Q^{-x}}{Q - Q^{-1}}. \tag{2.21}$$

To understand whether p and q are two genuinely independent quantization parameters, one must study the comultiplication rules for the (p, q)-Virasoro generators $L_n^{(\Delta)}$. For the present case of an arbitrary Δ ($\neq 0, 1$) we could not find the comultiplication rule. An exactly analogous situation, however, also develops in the $\Delta = 0, 1$ case discussed later. There, the structure of the corresponding comultiplication rule depends on both Q and λ. Consequently, p and q may be regarded as two independent deformation parameters. An understanding of the comultiplication rule for $L_n^{(\Delta)}$ for a generic Δ ($\neq 0, 1$) is, therefore, important.

Using a suitably deformed commutator (2.10) may be written in a more transparent form with numerical structure constants

$$[L_n^{(\Delta)}, L_m^{(\Delta)}]_{R_{nm}, S_{mn}} = [m - n]L_{n+m}^{(\Delta)} \tag{2.22}$$

where

$$R_{nm} = (q^{m-n} - p^{n-m})\chi_{nm}(q, p^{-1}) \tag{2.23}$$

$$S_{mn} = (q^{m-n} - p^{n-m})\chi_{mn}(p^{-1}, q) \tag{2.24}$$

$$\chi_{nm}(q, p^{-1}) = \left\{ q^{N_\Delta}\left(q^{-n} - \left(\frac{q}{p}\right)^{m-n} \frac{[m(\Delta-1)][\Delta n]}{[n(\Delta-1)][\Delta m]} q^{-m} \right) - p^{-N_\Delta}\left(p^{n} - \left(\frac{q}{p}\right)^{m-n} \frac{[m(\Delta-1)][\Delta n]}{[n(\Delta-1)][\Delta m]} p^{m} \right) \right\}^{-1}. \tag{2.25}$$

The numerical structure constants in (2.25) facilitate the construction of a (p, q)-deformed Jacobi identity. To this end, we use the identity

$$\left(\frac{q}{p}\right)^{-m} \frac{[2k]}{[k]} [m-k][m+n-k] + \text{cyclic permutations} = 0 \tag{2.26}$$

to establish

$$\left(\frac{q}{p}\right)^{-m} \frac{[2k]}{[k]} [L_k^{(\Delta)}, [L_n^{(\Delta)}, L_m^{(\Delta)}]_{R_{nm}, S_{mn}}]_{R_{k\,n+m}, S_{n+m\,k}} + \text{cyclic permutations} = 0. \tag{2.27}$$

The identity (2.27) may be used to search for a (p, q)-deformation of the central term of the Virasoro algebra. We assume a central extension of the algebra (2.22)

$$[\hat{L}_n^{(\Delta)}, \hat{L}_m^{(\Delta)}]_{R_{nm}, S_{mn}} = [m-n]\hat{L}_{n+m}^{(\Delta)} + \delta_{n+m,0}\hat{C}_n(q, p) \tag{2.28}$$

where the following property

$$[\hat{L}_k^{(\Delta)}, \hat{C}_n(q, p)]_{R_{k0}, S_{0k}} = 0 \tag{2.29}$$

is assumed to be valid. With further assumption of a factorization scheme

$$\hat{C}_n(q, p) = \hat{\Gamma}(N_\Delta) C_n(q, p) \tag{2.30}$$

where the entire n-dependence resides in the c-number term $C_n(q, p)$, we obtain from (2.27)-(2.30)

$$\left(\frac{q}{p}\right)^{-m} \frac{[2k]}{[k]} [m-n] C_k(q, p) \delta_{k+m+n,0} + \text{cyclic permutations} = 0. \tag{2.31}$$

The solution of (2.31) is

$$C_n(q, p) = C(q, p)\left(\frac{q}{p}\right)^{-2n} \frac{[n]}{[2n]} [n-1][n][n+1] \tag{2.32}$$

where $C(q, p)$ is an arbitrary function of (p, q); this possibility of dependence of $C(q, p)$ on p and q individually—not necessarily through $Q = \sqrt{pq}$ alone—makes p and q independent deformation parameters. In the limit $p = q$, the c-number term (2.32) reduces to the value obtained in [7]. Substituting (2.30) in (2.29), we obtain an equation for $\hat{\Gamma}(N_\Delta)$ for each Δ. We enlist some special cases:

$$(q/p)^{k/2} \hat{L}_k^{(\Delta)} \hat{\Gamma}(N_\Delta) - \hat{\Gamma}(N_\Delta) \hat{L}_k^{(\Delta)} = 0 \qquad (\text{for } \Delta = \tfrac{1}{2}) \tag{2.33}$$

$$\hat{L}_k^{(\Delta)} \hat{\Gamma}(N_\Delta) - \frac{p^k + q^{-k}}{2} \hat{\Gamma}(N_\Delta) \hat{L}_k^{(\Delta)} = 0 \qquad (\text{for } \Delta = 2) \text{ etc.} \tag{2.34}$$

The solution for (2.33) is immediate

$$\hat{\Gamma}(N_\Delta) = (q/p)^{N_\Delta/2}. \tag{2.35}$$

Therefore for the physically important case $\Delta=\frac{1}{2}$, corresponding to the energy density in the Ising model, we obtain from (2.30), (2.32) and (2.35) the full central charge term for the (p,q)-Virasoro algebra

$$\hat{C}_n^{\Delta=1/2}(q,p)=C(q,p)\left(\frac{q}{p}\right)^{N_\Delta/2-2n}\frac{[n]}{[2n]}[n-1][n][n+1]. \tag{2.36}$$

Notice that in the limit $p=q$, the term $\hat{C}_n^{\Delta=1/2}$ reduces to a c-number.

3. (p,q)-Virasoro algebra and (p,q)-KdV equation: $\Delta=0,1$ case

For $\Delta=0,1$ case we employ (2.9) to obtain the product rule

$$L_n^{(\Delta)}L_m^{(\Delta)}=[z\partial_z+\Delta-n]L_{n+m}^{(\Delta)} \tag{3.1}$$

which yields the following closed algebra

$$[L_n^{(\Delta)},L_m^{(\Delta)}]_{x,y}=(q-p^{-1})^{-1}\{q^{N_\Delta}(xq^{-n}-yq^{-m})-p^{-N_\Delta}(xp^n-yp^m)\}L_{n+m}^{(\Delta)} \tag{3.2}$$

for the arbitrary numbers x, y. For a special choice

$$x=1 \qquad y=q^{m-n} \tag{3.3}$$

(3.2) reduces to the form

$$[L_n^{(\Delta)},L_m^{(\Delta)}]_{1,q^{m-n}}=[m-n]p^{-N_\Delta+m}L_{n+m}^{(\Delta)}. \tag{3.4}$$

Redefining the generators

$$\tilde{L}_n^{(\Delta)}=p^{N_\Delta}L_n^{(\Delta)} \tag{3.5}$$

we obtain the following relations

$$[\tilde{L}_n^{(\Delta)},\tilde{L}_m^{(\Delta)}]_{p^{n-m},q^{m-n}}=[m-n]\tilde{L}_{n+m}^{(\Delta)} \tag{3.6}$$

$$[\tilde{L}_n^{(\Delta)},\tilde{L}_m^{(\Delta)}]=[m-n]p^{N_\Delta-n}q^{N_\Delta-m}\tilde{L}_{n+m}^{(\Delta)}. \tag{3.7}$$

Using the symmetry $q\leftrightarrow p^{-1}$, we may obtain another set of complimentary relations for (3.4)–(3.7). From (3.6)–(3.7) we notice an $\mathrm{su}_{p,q}(1,1)$ subalgebra

$$\begin{aligned}&[\tilde{L}_0^{(\Delta)},\tilde{L}_1^{(\Delta)}]_{p^{-1},q}=\tilde{L}_1^{(\Delta)} \qquad [\tilde{L}_{-1}^{(\Delta)},\tilde{L}_0^{(\Delta)}]_{p^{-1},q}=\tilde{L}_{-1}^{(\Delta)}\\&[\tilde{L}_{-1}^{(\Delta)},\tilde{L}_1^{(\Delta)}]=[2]q^{-1}p(\tilde{L}_0^{(\Delta)}+(q-p^{-1})\tilde{L}_0^{(\Delta)2}).\end{aligned} \tag{3.8}$$

We may consider (3.8) as a (p,q)-generalization of the deformation considered by Witten [18] in the context of the vertex models.

Using the redefined generators

$$\tilde{\mathscr{L}}_n^{(\Delta)}=\lambda^{-1}\tilde{L}_n^{(\Delta)} \tag{3.9}$$

the algebra (3.6) may be reduced to the well known [1-3] Q-deformed Virasoro algebra

$$[\tilde{\mathscr{L}}_n^{(\Delta)},\tilde{\mathscr{L}}_m^{(\Delta)}]_{Q^{n-m},Q^{m-n}}=[m-n]_Q\tilde{\mathscr{L}}_{n+m}^{(\Delta)}. \tag{3.10}$$

However, as stressed by Schirrmacher *et al* [12] in the context of $\mathrm{GL}_{p,q}(2)$, whether p and q are to be treated as two independent deformation parameters must be settled by examining the comultiplication rules for the generators $\tilde{L}_n^{(\Delta)}$. Considering a continuum formulation for algebra (3.10) Devchand *et al* [6] constructed a comultiplication rule for the corresponding generating functional. We extend their result to obtain a comultiplication rule for the generators of the (p,q)-Virasoro algebra.

In the continuum formulation [6], a single generating functional $X(\phi)$ replaces $\tilde{L}_n^{(\Delta)}$ and is thought to be acting on ϕ which is an element of an associative and commutative algebra E. The (p, q)-deformed algebra (3.6) in this construction has an arbitrary parameter α and is of the form

$$X(p^{\partial}\phi)X(p^{-\partial}\psi)-X(q^{\partial}\psi)X(q^{-\partial}\phi) = X\left((p^{\partial+\alpha}\phi)([\partial+\alpha]\psi)-\left(\left(\frac{p}{q}\right)^{\partial+\alpha}[\partial+\alpha]\phi\right)(q^{\partial+\alpha}\psi)\right) \tag{3.11}$$

as may be seen by a Fourier decomposition.

The derivative ∂ acts on the algebra E and may be replaced in (3.11) with $D_{p,q}$ defined in (2.8), provided we choose the (p, q)-exponentials

$$\exp_{p,q}(x)=\sum_{n=0}^{\infty}\frac{x^n}{[n]!} \tag{3.12}$$

as the basis functions in the Fourier expansion of (3.11). The comultiplication rule of the functional generator reads

$$\Delta X(\phi)=X(\Lambda_1\phi)\otimes\mathbb{1}+\mathbb{1}\otimes X(\Lambda_2\phi) \tag{3.13}$$

where $\Lambda_{1,2}(p, q)$ are the operators acting on the algebra E and to be determined by requiring that ΔX satisfies (3.11). Using the later criterion we obtain

$$\begin{aligned}
&(X(\Lambda_1p^{\partial}\phi)X(\Lambda_1p^{-\partial}\psi)-X(\Lambda_1q^{\partial}\psi)X(\Lambda_1q^{-\partial}\phi))\otimes\mathbb{1}\\
&\quad+\mathbb{1}\otimes(X(\Lambda_2p^{\partial}\phi)X(\Lambda_2p^{-\partial}\psi)-X(\Lambda_2q^{\partial}\psi)X(\Lambda_2q^{-\partial}\phi))\\
&\quad+(X(\Lambda_1p^{\partial}\phi)\otimes X(\Lambda_2p^{-\partial}\psi)-X(\Lambda_1q^{-\partial}\phi)\otimes X(\Lambda_2q^{\partial}\psi))\\
&\quad+(X(\Lambda_1p^{-\partial}\psi)\otimes X(\Lambda_2p^{\partial}\phi)-X(\Lambda_1q^{\partial}\psi)\otimes X(\Lambda_2q^{-\partial}\phi))\\
&=X\left((p^{\partial+\alpha}\Lambda_1\phi)([\partial+\alpha]\Lambda_1\psi)-\left(\left(\frac{p}{q}\right)^{\partial+\alpha}[\partial+\alpha]\Lambda_1\phi\right)(q^{\partial+\alpha}\Lambda_1\psi)\right)\otimes\mathbb{1}\\
&\quad+\mathbb{1}\otimes X\left((p^{\partial+\alpha}\Lambda_2\phi)([\partial+\alpha]\Lambda_2\psi)-\left(\left(\frac{p}{q}\right)^{\partial+\alpha}[\partial+\alpha]\Lambda_2\phi\right)(q^{\partial+\alpha}\Lambda_2\psi)\right).
\end{aligned} \tag{3.14}$$

From (3.11) it appears that (3.14) is satisfied provided

$$\Lambda_i(p^{\partial}\mp q^{-\partial})\phi=0 \qquad \text{for } i=1,2 \qquad \text{and} \qquad q\neq p^{-1}. \tag{3.15}$$

A formal solution for (3.15) is

$$\Lambda_i=C_0^i\sum_{n\in\mathbb{Z}}(\pm1)^n(pq)^{n\partial}+C_1^i\sum_{n\in\mathbb{Z}}(\pm1)^n p^{(n+1)\partial}q^{n\partial}+C_{-1}^i\sum_{n\in\mathbb{Z}}(\pm1)^n p^{n\partial}q^{(n+1)\partial}. \tag{3.16}$$

The point we stress here is that the algebra (3.6) has two independent deformation parameters (p, q) as the comultiplication rule (3.13), (3.16) depends individually on these parameters. We also notice that for the choice of the negative sign in (3.15), the parameter α is to be chosen non-zero in order to make the RHS of (3.14) non-vanishing.

The centrally extended version of the algebra (3.6) reads

$$[\hat{L}_n^{(\Delta)}, \hat{L}_m^{(\Delta)}]_{p^{n-m},q^{m-n}}=[m-n]\hat{L}_{n+m}^{(\Delta)}+\delta_{n+m,0}\hat{C}_n(q, p). \tag{3.17}$$

Following a procedure parallel to the discussion in section 2 and assuming

$$[\hat{L}_k^{(\Delta)}, \hat{C}_n(q,p)]_{p^k, q^{-k}} = 0 \tag{3.18}$$

we obtain a solution for $\hat{C}_n(q,p)$ as

$$\hat{C}_n(q,p) = C(q,p) q^{N_\Delta - 2n} p^{N_\Delta + 2n} \frac{[n]}{[2n]} [n-1][n][n+1]. \tag{3.19}$$

Introducing the generator

$$\hat{\hat{L}}_n^{(\Delta)} = (qp)^{-N_\Delta} \hat{L}_n^{(\Delta)} \tag{3.20}$$

we rewrite the algebra (3.17) as a commutation relation

$$[\hat{\hat{L}}_n^{(\Delta)}, \hat{\hat{L}}_m^{(\Delta)}] = [m-n] q^{-N_\Delta + n} p^{-N_\Delta + m} \hat{\hat{L}}_{n+m}^{(\Delta)} + C(q,p) \delta_{n+m,0} q^{-N_\Delta - n} p^{-N_\Delta + n} \frac{[n]}{[2n]} [n-1][n][n+1]. \tag{3.21}$$

As a physical application of the deformed (p, q)-Virasoro algebra discussed earlier, we will now study the correspondence between the algebra (3.21) and a (p, q)-KdV equation. The well known [16, 17] interrelation between the Virasoro algebra and the usual KdV equation was utilized by Chaichian *et al* [4] to discover a q-deformed KdV equation corresponding to q-Virasoro algebra. Their method hinges on constructing a current which defines a bi-Hamiltonian structure and thereby satisfies a nonlinear evolution equation. We follow this technique to describe a (p, q)-KdV structure corresponding to the algebra (3.21).

To this end, we define a current

$$u(\sigma) = \sum_{n \in \mathbf{Z}} \hat{\hat{L}}_n^{(\Delta)} \, \mathrm{e}^{-\mathrm{i}n\sigma}. \tag{3.22}$$

Using the parametrization (2.16), we obtain the commutator

$$\frac{1}{2\pi \mathrm{i}} [u(\sigma), u(\sigma')] = \frac{\lambda}{2 \sin \varepsilon} (\mathrm{e}^{-2\varepsilon \partial_\sigma} u(\sigma) - u(\sigma) \mathrm{e}^{2\varepsilon\partial_\sigma}) Q^{-2N_\Delta} \delta(\sigma - \sigma') - C\lambda^3 \frac{\sinh \varepsilon \partial_\sigma}{\sinh 2\varepsilon\partial_\sigma} \frac{\sinh \varepsilon(\partial_\sigma + \mathrm{i}) \sinh \varepsilon \partial_\sigma \sinh \varepsilon(\partial_\sigma - \mathrm{i})}{\sin^3 \varepsilon} Q^{-2N_\Delta} \delta(\sigma - \sigma') \tag{3.23}$$

where

$$Q = \mathrm{e}^{-\mathrm{i}\varepsilon}. \tag{3.24}$$

The commutator (3.23) admits [4] a Hamiltonian system defined by

$$H_2 = \frac{1}{4\pi} \int_0^{2\pi} \mathrm{d}\sigma \, u^2(\sigma) \tag{3.25}$$

which yields a (p, q)-deformed KdV equation

$$\dot{u} = \frac{\lambda}{4 \sin \varepsilon} (\mathrm{e}^{-2\varepsilon\partial_\sigma} u(\sigma) - u(\sigma) \mathrm{e}^{2\varepsilon\partial_\sigma})(Q^{-2N_\Delta} u(\sigma) + u(\sigma) Q^{-2N_\Delta}) - C \frac{\lambda^3}{2} \frac{\sinh \varepsilon\partial_\sigma}{\sinh 2\varepsilon\partial_\sigma} \frac{\sinh \varepsilon(\partial_\sigma + \mathrm{i}) \sinh \varepsilon\partial_\sigma \sinh \varepsilon(\partial_\sigma - \mathrm{i})}{\sin^3 \varepsilon} \times (Q^{-2N_\Delta} u(\sigma) + u(\sigma) Q^{-2N_\Delta}). \tag{3.26}$$

The different λ-dependences of the central and the non-central terms in the RHS of (3.26), in contradistinction to the single parameter case, and the dependence of C on both Q and λ should be useful in applications. In the undeformed limit ($Q=1$, $\lambda=1$) we obtain the usual KdV equation.

4. Conclusion

In summary, we have obtained a Δ-dependent differential representation of a deformed (p, q)-Virasoro algebra. In the $\Delta=0, 1$ case we have constructed, for the generating functional, a comultiplication rule depending individually on both parameters. A central charge for the (p, q)-Virasoro algebra has been described as a consequence of a (p, q)-deformed Jacobi identity. For $\Delta=0, 1$ case we have obtained a (p, q)-deformed KdV equation.

Note added in proof. In the case of a centreless Virasoro algebra further generalization of the deformation, based on the oscillator realization, has also been discussed recently [19].

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